Question 14 **(11 marks)**

A manufacturer of chocolate produces 3 times as many soft centred chocolates as hard centred ones. The chocolates are randomly packed in boxes of 20.

(a) Find the probability that in a box there are

(i) an equal number of soft centred and hard centred chocolates (3 marks)

> Let the rv X be the number of hard centred chocolates per box of 20.
>
> Then $X \sim \text{Bin}(20, 0.25)$
>
> $P(X = 10) = 0.00992$

(ii) fewer than 5 hard centred chocolates. (1 mark)

> $P(X < 5) = P(X \leq 4) = 0.41484$

(b) Determine the mean and standard deviation of the number of hard centred chocolates in a box of 20. (2 marks)

> Mean: $np = 20 \times 0.25 = 5$
>
> SD: $\sqrt{np(1-p)} = \sqrt{20 \times 0.25 \times 0.75} = \sqrt{3.75} = 1.9365$

(c) A random sample of 5 boxes is taken from the production line. Find the probability that exactly 3 of them contain fewer than 5 hard centred chocolates. (2 marks)

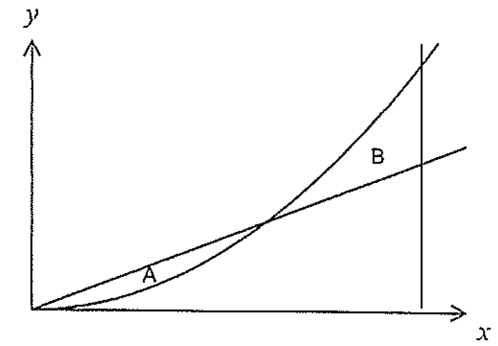
> Let the rv Y be the number of boxes out of 5 with fewer than 5 hard centres.
>
> Then $Y \sim \text{Bin}(5, 0.41484)$
>
> $P(Y = 3) = 0.24445$

(d) A random sample of 30 boxes is taken from the production line. Find the probability that the mean number of hard centred chocolates per box in the sample exceeds 5.5. (3 marks)

> For samples of size 30 or more, CLT says distribution of sample means $\bar{X}$ will be approximately normally distributed as follows:
>
> $\bar{X} \sim N\left(5, \frac{1.9365^2}{30}\right) \sim N(5, 0.35355^2)$
>
> Hence $P(\bar{X} > 5.5) = 0.07865$

Question 19 **(7 marks)**

The graph below, not to scale, shows the functions $f(x)=\frac{x}{10}$, $g(x)=\frac{x^2}{10}$ and the line $x=2$.



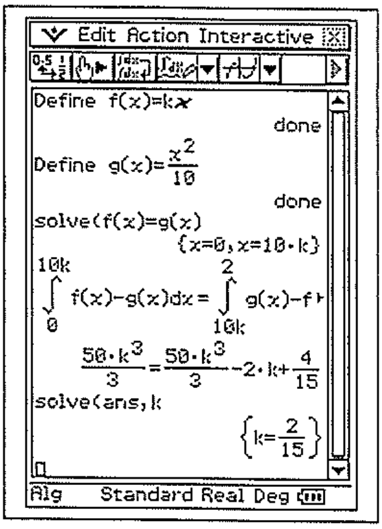
Region A is the area trapped by f and g.
Region B is the area trapped by f, g and the line $x=2$.

(a) Find the areas of regions A and B. (3 marks)

> f and g intersect when $x=1$.
>
> Region A = $\int_0^1 (f-g)\,dx = \frac{1}{60}$
>
> Region B = $\int_1^2 (g-f)\,dx = \frac{1}{12}$

(b) $f(x)$ is modified to become the line $f(x)=kx$, so that the area of region A is exactly the same as the area of region B. Determine the value of k. (4 marks)

> f and g intersect when $x=10k$.
>
> $\int_0^{10k} (f-g)\,dx = \int_{10k}^{2} (g-f)\,dx$
>
> $\frac{50k^3}{3} = \frac{50k^3}{3} - 2k + \frac{4}{15}$
>
> $k = \frac{2}{15}$



Question 11 **(6 marks)**

A body moves in a straight line so that its displacement, $x(t)$ metres, from a fixed point after t seconds is given by $x(t) = t^3 - 9t^2 + 24t$, for $0 \le t \le 5$.

(a) When is the body stationary? (2 marks)

$v(t) = 3t^2 - 18t + 24$
$= 0$
when $t = 2$ or $t = 4$.

(b) When is the body moving fastest? (2 marks)

$v(0) = 24$
$v(5) = 9$
$v(3) = -3$ (Turning point)

Hence fastest when $t = 0$

(c) Calculate the distance travelled by the body in the first four seconds. (2 marks)

Either

$$\int_0^2 v(t)\,dt + \left|\int_2^4 v(t)\,dt\right| = 20 + |-4| = 24 \text{ metres}$$

Or

$$\int_0^4 |v(t)|\,dt = 24$$

(3)

Question 4 **[16 marks]**

(a) Simplify the following expression giving your answer with positive indices.

$$-2a^3b^{-1} \times \left(\frac{3a^2}{b}\right)^2 \qquad [4]$$

$$= -2a^3b^{-3} \times 9a^4b^{-2}$$
$$= -18a^7b^{-3}$$
$$= \frac{-18a^7}{b^3}$$

(b) Solve to find x for $\log_3 1 = x$ [1]

$$x = 0$$

(c) Using log rules, solve $5^{x+1} = 4$ giving x as an exact answer. [3]

$$\ln 5^{x+1} = \ln 4$$
$$(x+1)\ln 5 = \ln 4$$
$$x + 1 = \frac{\ln 4}{\ln 5}$$
$$x = \frac{\ln 4}{\ln 5} - 1 \quad or \quad \frac{\ln 4 - \ln 5}{\ln 5}$$

(d) Find an exact solution $\ln(x-3) = 2$

Take the e of both sides $(x-3) = e^2$

$$x = e^2 + 3$$

(4)

(e) Solve to find x for $\log_3\left(\frac{2x+7}{x}\right)=2$ (Hint: Use the power equivalent form) [3]

$$2^3=\frac{2x+7}{x}$$
$$8x=2x+7$$
$$6x=7$$
$$x=\frac{7}{6}$$

(f) Solve for x using index laws $\frac{5}{\sqrt[3]{x^2}}= 80$ [3]

$$5x^{-\frac{2}{3}}=80$$
$$x^{-\frac{2}{3}}=16$$
$$x^{\frac{2}{3}}=\frac{1}{16}$$
$$x=\frac{1}{16^{\frac{3}{2}}}=\frac{1}{64}$$

5

Question 7 **[6 marks]**

An Object moves along a straight line so that its velocity (m/s) at time t seconds, is given by:

$$v = 50 + 5t - t^2$$

(a) Find the displacement x after 10 secs if initially the object is 20 metres in the positive direction from the fixed point. [4]

$$x = \int 50t + 5t - t^2 dt = 50t + \frac{5t^2}{2} - \frac{t^3}{3} + c$$

if $x = 20$ when $t = 0$ then $c = 20$

$$x = 50t + \frac{5t^2}{2} - \frac{t^3}{3} + 20$$

Therefore $x(10) = 50 \times 10 + \frac{5 \times 10^2}{2} - \frac{10^3}{3} + 20 = 436\frac{2}{3} m$

(b) Find t when there is no acceleration. [2]

$a = 5 - 2t$

When $t = 0$, $a = 5\text{m/s}^2$

Question 8 **[5 Marks]**

$$5^x = e^{x-2}$$

(a) By using the solve facility on your calculator, solve for x, giving your answer to 3 decimal places. [1]

$x = -3.282$

(b) Now prove using ln rules that the exact answer is $x = \frac{-2}{Ln5 \ \ -1}$ [4]

$$\ln 5^x = x - 2$$
$$x \ln 5 = x - 2$$
$$x \ln 5 - x = -2$$
$$x(\ln 5 - 1) = -2$$
$$x = \frac{-2}{\ln 5 - 1}$$

(6)

Question 16 **[5 marks]**

a. Find the exact answer for $\int_1^5 \frac{1}{x}\ dx$ [2]

$$\int_1^5 \frac{1}{x}\ dx = \left[\ln|x|\right]_1^5 = \ln 5 - \ln 1 = \ln 5$$

b. Show that the exact answer for $\int_{-6}^{-4} \frac{1}{x+1}\ dx$ equals $Ln\left(\frac{3}{5}\right)$ [3]

$$\int_{-6}^{-4} \frac{1}{x+1}\ dx = \left[\ln|x+1|\right]_{-6}^{-4} = \ln|-3| - \ln|-5|$$

$$= \ln 3 - \ln 5 = \ln\left(\frac{3}{5}\right)$$

7

Question 17 [8 marks]

During a nuclear accident, an amount of a radioactive isotope is released. It decays at a rate of 1% a week.

(a) Determine the half life of this radioactive material. [2]

$0.5R_0 = R_0 \times e^{-0.01 \times t}$

$0.5 = e^{-0.01t}$

$t = 69.3$ weeks

(b) What % of the original amount is present after 4 weeks. [3]

$R = R_0 \times e^{-0.04}$

$\frac{R}{R_0} = 0.96$

96%

(c) ~~(d)~~ Another element P is also released. After monitoring it for a few weeks, scientists concluded the amount present (kg) fits the curve

$$A_p = \frac{35Ln\ (t+1)}{t}$$

When will the 2 elements have the same quantity of material present if 20 kg of the first element was released? (to the nearest minute) [3]

$$20 \times e^{-0.01 \cdot t} = \frac{35Ln\ (t+1)}{t}$$

t = 1.8972215 weeks

t = 19124 mins = 1week, 6days, 6hour 44mins

(8)

The end

Section One: Calculator-free **(50 Marks)**

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(6 marks)**

Differentiate the following with respect to x. Do not simplify your answer.

(a) $y = (1 + \frac{2}{x^2})^3$ (3 marks)

$y' = 3\left(1 + \frac{2}{x^2}\right)^2 \times \left(-4x^{-3}\right)$ ✓ ✓ ✓

(b) $y = x^2 e^{2x}$ (3 marks)

$y' = 2x e^{2x} + x^2 (2) e^{2x}$ ✓ ✓ ✓

(9)

Question 3 **(7 marks)**

(a) Express y in terms of x free of logarithms if $2\log_e x+1=\frac{\log_e 3y}{2}$. (3 marks)

$4\log_e x + 2 = \log_e 3y$

$\log_e x^4 + \log_e e^2 = \log_e 3y$ ✓

$\log_e x^4 e^2 = \log_e 3y$ ✓

$y = \frac{x^4 e^2}{3}$ ✓

(b) Solve $4^x - 3(2^x) + 2 = 0$. (4 marks)

(Hint: Let $y = 2^x$).

$y^2 - 3y + 2 = 0$ ✓

$(y-2)(y-1) = 0$

$y = 1$ or 2 ✓

$\therefore 2^x = 1$ or $2^x = 2$ ✓

$\therefore x = 1$ or 0. ✓

10

Question 2 **(4 marks)**

Differentiate the following, without simplifying:

(a) $y = \dfrac{3}{\sqrt{1+e^{5x}}}$ (2 marks)

$y = 3(1+e^{5x})^{-0.5}$

$y' = 3(-0.5)(5e^{5x})(1+e^{5x})^{-1.5}$

(b) $y = \dfrac{x^3-4}{x-2}$ (2 marks)

$u = x^3 - 4 \qquad v = x - 2$

$u' = 3x^2 \qquad v' = 1$

$y' = \dfrac{3x^2(x-2) - 1(x^3-4)}{(x-2)^2}$

Question 3 **(4 marks)**

Determine the domain and range of $f \circ g(x)$, where $f(x) = 2^{x+2}$ and $g(x) = \sqrt{x+1}$.

$f \circ g(x) = f(\sqrt{x+1})$

$= 2^{\sqrt{x+1}+2}$

Domain: $x + 1 \geq 0 \Rightarrow x \geq -1$.

Range: $y \geq 2^2 \Rightarrow y \geq 4$.

Question 5 **(5 marks)**

Determine the following integrals:

(a) $\int (6x+9)\left(3x+x^2\right)^2 dx$ (2 marks)

$$= 3\int (3+2x)\left(3x+x^2\right)^2 dx$$
$$= 3\frac{\left(3x+x^2\right)^3}{3} + c$$
$$= \left(3x+x^2\right)^3 + c$$

(b) $\int_1^4 3\sqrt{x}\, dx$ (3 marks)

$$= \left[\frac{3x^{3/2}}{3/2}\right]_1^4$$
$$= \left[2(x)^{3/2}\right]_1^4$$
$$= 2(4)^{3/2} - 2(1)^{3/2}$$
$$= 16 - 2$$
$$= 14$$

12

Question 6 **(4 marks)**

The volume, V in cm^3, of an object is changing with time, t in seconds, so that the volume at any time is given by $V = 5t + \frac{12}{t}$. Use the incremental formula to find the approximate change in volume of the object between $t = 2$ and $t = 2.01$ seconds.

For small change, $\partial V = \frac{dV}{dt}\partial t$.

$$\frac{dV}{dt} = 5 - \frac{12}{t^2}$$

$$\begin{aligned}\partial V &= \left(5 - \frac{12}{t^2}\right)\partial t \\ &= \left(5 - \frac{12}{2^2}\right) \times (2.01 - 2) \\ &= 2 \times 0.01 \\ &= 0.02 \text{ cm}^3\end{aligned}$$

13

Question 8 **(7 marks)**

Determine the coordinates of all roots, stationary points and points of inflection of the function $y = x^3(4+x)$. Justify the nature of the stationary points found using a standard test.

Roots:
$x^3 = 0$ or $4 + x = 0$.
Hence roots at (0, 0) and (-4, 0).

Stationary points:
$y = 4x^3 + x^4$
$\frac{dy}{dx} = 12x^2 - 4x^3$
$= 4x^2(3 - x)$
$\therefore x^2 = 0$ or $3 - x = 0$
Hence stationary points at (0, 0) and (-3, -27).

Nature:
$\frac{d^2y}{dx^2} = 24x + 12x^2$
$= 12x(2 + x)$
$y''(0) = 0 \Rightarrow$ Horizontal pt of inflection (as $y'(0) = 0$)
$y''(-3) = +\text{ve} \Rightarrow$ Minimum
Hence horizontal point of inflection at (0, 0) and minimum at (-3, -27).

Points of inflection:
$12x(2 + x) = 0$
when $x = 0$ or $x = -2$.
(0, 0) already stated above, another point of inflection at (-2, -16).

(14)

End of questions

Question 11 **(10 marks)**

Caesium-137 is a common radioisotope used as a gamma-emitter in industrial applications and it has a half-life (the time it takes half of the mass to decay) of 30.17 years. In an application involving sterilization of medical instruments, the mass (M in milligrams) of Caesium-137 used in the system decays according to the relationship:

$$M = \frac{(2.7)^{kt}}{5}$$

where k is a constant and t is the time (in years) since the system was manufactured.

There are 1 000 micrograms in 1 milligram.

(a) What mass of Caesium-137 was initially in the system, in micrograms? (2 marks)

$M(0) = \frac{2.7^0}{5}$ ✓

$= 0.2 = 200$ micrograms ✓

(b) Calculate the value of k, rounding your answer to 3 significant figures. (3 mark)

$0.1 = \frac{2.7^{30.17K}}{5}$ ✓

$K = -0.0231$ ✓

(c) What percentage of the original mass of Caesium-137 remains in the system after one year? (2 marks)

$M = \frac{2.7^{-0.0231(1)}}{5} = 0.1955$ ✓

$\therefore \% = \frac{0.1955}{0.2} \times 100 = 97.75\%$ ✓

(d) The Caesium-137 in the sterilization system is replaced once its initial mass has fallen by more than 40 micrograms. After how many years and months would this be necessary? (3 marks)

$200 - 40 = 160$ mircograms $= 0.16$ mg

$0.16 = \frac{2.7^{-0.0231t}}{5}$ ✓

$t = 9.7255$ yrs ✓

$= 9$ yrs 9 mths. ✓

 (15)

Section Two: Calculator-assumed **(80 Marks)**

This section has **twelve (12)** questions. Answer **all** questions. Write your answers in the space provided.

Working time for this section is 100 minutes.

Question 9 **(4 marks)**

The percentage of trees, P, in a plantation affected by a disease was changing with time, t in months, according to the relationship $\frac{dP}{dt} = -0.017P$.

(a) Was the health of the plantation getting better or worse? Briefly justify your answer by referring to the above relationship. (1 mark)

> Better, as the rate of change of diseased trees with time is negative (-0.017) and hence decreasing.

(b) If 7.2% of the trees in the plantation were affected today, what percentage is expected to be affected by the disease in one and a half years time? (3 marks)

> $t = 18$ months
> $P = 7.2e^{-0.017 \times 18}$
> $= 5.3\%$

16

Question 20 **(5 marks)**

A random variable X is normally distributed such that the mean is twice the variance and the probability that X is greater than 21.5 is 0.231. Find the mean and standard deviation of X.

Let x = standard deviation and y = mean.

$y = 2x^2$ (1)

$P(Z > z) = 0.231 \Rightarrow z = 0.73556$

$\frac{21.5 - y}{x} = 0.73556$ (2)

Solve 1 & 2 simultaneously to get

$x = 3.099982$

$y = 19.2197$

(Ignore other solution as standard deviation must be positive)

Hence mean is 19.22 and standard deviation is 3.10 (2dp).

End of questions